LÈVY PROCESSES AND BALAYAGE SPACE

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Abstract: Bliedtner, Hansen [2] state the theorem which gives the characterization for the cones of excessive functions for sub-Markov processes, where all excessive functions are lower semicontinuous and positive hyperharmonic functions for a family of harmonic kernels on X. We formulate the consequence of this rather general theorem for Lévy processes.

Abstrakt: Bliedtner, Hansen [2] uvádějí obecný theorem, který umožňuje charakterizovat kužel excesivních funkcí pro sub-markovské procesy tak, že všechny tyto excesivní funkce jsou zdola polospojité a kladné hyperharmonické funkce pro rodinu harmonických jader na lokálně kompaktním prostoru X. V tomto článku vysvětlíme výše uvedené pojmy, vyslovíme zmíněnou větu a formulujeme důsledek pro Lèvyho procesy.

1 Introduction

The classical potential theory provides a very close connection between theory of probability and mathematical calculus. A very known is a powerfull Feymann-Kac theorem which very formally speaking allows us to interpret the solution of the partial differential equation of a specific form as an expected value of the functional of Brownian motion stopped at a certain time. The link between these two branches of mathematics is however much wider. We can look at potential theory from four different angles via Hunt processes, sub-Markov semigroups, families of harmonic kernels and balayage space. This is stated in Theorem IV.8.1 in Bliedtner, Hansen [2]).

Our motivation comes from the interest in the family of functions which solves generalized Dirichlet problem. Consider for example a heat equation. The functions which are harmonic with respect to the heat equation are a special example of the general class of harmonic kernels. The harmonic functions are also closely related with the class of Brownian semigroups. Our intention is to study Lèvy processes. The later covers large class of stochastic processes which however still owns very nice and handable properties.

The article is organized as follows. We firstly introduce notation and give definitions of balayage space, sub-Markov semigroups, excessive functions, hyperharmonic functions, family of harmonic kernels and Hunt processes in the section Preliminaries. The reader familiar with these terms may prefer to skip this part. In the second section we state a rather general theorem from Bliedtner J., Hansen W. [2]. In the last section we recall the definition of Lèvy process and use the theorem from the second part to formulate a consequence of it for Lèvy processes as a special case of Hunt processes.

The introduced theory is of great importance not only for the theoretical studies. It has great applicability to optimal stopping problems and stochastic control problems and consequently is of interest e.g. in the pricing of the American contingent claims. For an introduction to the latter topics and Dirichlet problem the reader may appriciate chapters 9-12 from Oksendal [3].

2 Preliminaries

We consider a locally compact space X with a countable base and denote $\mathcal{U}$ a base for the topology on X consisting of relatively compact open sets in X. We further denote by $\mathcal{C}(X)$ space of all continuous real functions on X and $\mathcal{C}_0(X)$ space of all continuous real functions vanishing in infinity. We also fix the probability space $(\Omega, \mathcal{F}, P)$ on which live considered stochastic processes.

In the following we explain what is a balayage space. Let us consider $\mathcal{W}$ a convex cone of positive lower semicontinuous[1] numerical functions on X. For $v \in \mathcal{W}$ we denote $\hat{v}$ the lower regularization of v, i.e. $\hat{v}(x) := \liminf_{y \to x} v(y)$, $x \in X$.

The coarsest[2] topology which is at least as fine as the initial topology and for which all the functions from $\mathcal{W}$ are continuous will be called $(\mathcal{W})$-fine topology.

Observe that functions from a convex cone $\mathcal{W}$ are missing the continuity property in the initial topology $\mathcal{U}$ of locally relatively compact open sets and thus we take new topology such that we add some sets to topology $\mathcal{U}$ in which the functions from $\mathcal{W}$ will be continuous.

By $\hat{v}^f$ we will denote the lower regularization of the function v in the $\mathcal{W}$-fine topology.

Definition 1. $(X, \mathcal{W})$ is a balayage space, if the following properties are satisfied:

1. $\mathcal{W}$ is $\sigma-$stable, i.e. for every increasing sequence (v_n) of functions from $\mathcal{W}$ the supremum $\sup v_n$ is in $\mathcal{W}$
2. $\widehat{\inf \mathcal{V}}^f \in \mathcal{W}$ for every subset $\mathcal{V}$ of $\mathcal{W}$.
3. if $u, v', v'' \in \mathcal{W}$ such that $u \leq v' + v''$, then there exists $u', u'' \in \mathcal{W}$ such that $u = u' + u''$ and $u' \leq v', u'' \leq v''$.
4. $\mathcal{W}$ is linearly separating, i.e. $\forall x, y \in X, x \neq y$ and $\lambda \geq 0$ exists $v \in \mathcal{W}$, s.t. $v(x) \neq \lambda v(y)$

[1]Real function f defined on the topological space is lower semicontinous if for every real α is the set $\{x; f(x) > \alpha\}$ open.

[2]Consider two topologies Θ_1, Θ_2 on X, s.t. $\Theta_1 \subseteq \Theta_2$ then we say that Θ_2 is finer then Θ_1 and Θ_1 is coarser then Θ_2.

and there exist positive $u_0, v_0 \in \mathcal{W} \cap \mathcal{C}(X)$ s.t. $\frac{u_0}{v_0} \in \mathcal{C}_0(X)$
and $v = \sup\{u \in \mathcal{W} \cap \mathcal{C}(X) : u \leq v\}$ for every $v \in \mathcal{W}$

As an example consider X a discrete space (at most countable) and $\mathcal{W}$ the set of all positive numerical functions on X, then $(X, \mathcal{W})$ is a balayage space.

Just for completness recall what we understand by the sub-Markov (resp. Markov) semigroup. We also explain what are excessive functions.

Definition 2. A family $\mathbb{P} = (P_t)_{t>0}$ of kernels[3] on X is a semigroup, if $P_{s+t} = P_s P_t$ for $s, t > 0$.
It is a sub-Markov (resp. Markov), if for every $t > 0$ $P_t \mathbb{I} \leq \mathbb{I}$ (resp.$P_t \mathbb{I} = \mathbb{I}$).

Further note, that having define a semigroup of Markov kernels we will call a positive borel measurable function $f \in \mathcal{B}^+(X)$ excessive with respect to the Markov semigroup $\mathbb{P}$ ($\mathbb{P}$-excessive) if $\sup_{t>0} P_t f = f$. The set of all $\mathbb{P}$-excessive functions we denote by $E_{\mathbb{P}} := \{f \in \mathcal{B}^+(X) : \sup_{t>0} P_t f = f\}$.

Now, we describe what are families of harmonic kernels and give definitions of hyperharmonic and superharmonic functions.

Consider a family of kernels $(H_U)_{U \in \mathcal{U}}$ on X. We call $(H_U)_{U \in \mathcal{U}}$ a family of sweeping kernels (relative to U) if for every set U from the base $\mathcal{U}$ of X, we have a kernel H_U on X for which holds that $H_U(x, U) = 0$ for every $x \in U$ and for every $x \in U^C :\ H_U(x, \cdot) = \delta_x$ where δ_x denotes dirac mass.

Now for every open subset V of X we denote $\mathcal{U}(V)$ the set of all open subsets $W \subseteq V$ such that closure $\bar{W}$ is a compact set in V. Let $\mathcal{H}^*(V)$ denote the set of all positive hyperharmonic functions on V, that is the set of all positive borel measurable functions v on X, such that v is lower semicontinuous on V and $-\infty < H_U v \leq v$ for all $U \subset \mathcal{U}(V)$. We also denote by $\mathcal{S}^*(V)$ the set of all positive superharmonic functions on V, i.e. the set of all positive hyperharmonic functions w on V for which $H_U w|_U \in \mathcal{C}(U)$ for all $U \subset \mathcal{U}(V)$.

We call $(H_U)_{U \in \mathcal{U}}$ a family of harmonic kernels if the following axioms are fullfiled:

1. $\forall x \in X, \lim_{U \downarrow x} H_U \mathbb{I}(x) = \mathbb{I}$
2. $H_V H_U = H_U$ for all $U, V \in \mathcal{U}$ and $\bar{V} \subset U$
3. For all $U \in \mathcal{U}$ and $f \in \mathcal{B}(X)$ which are bounded on ∂U the function $H_U f$ is continuous on U.
4. for $U \in \mathcal{U}$ and every $x \in U$ there exists a hyperharmonic function w s.t. $w(x) < \infty$ and $\lim_{\mathcal{A}} = \infty$ for every non-regular ultrafilter[4] $\mathcal{A}$ on U

[3] A kernel K on X is a mapping $K : X \times \mathcal{B}(X) \to \mathbb{R}^+$ s.t. $x \to K(x, B)$ is Borel measurable for every $B \in \mathcal{B}(X)$ and $B \to K(x, B)$ is a measure on $(X, \mathcal{B}(X))$ for every $x \in X$

[4] For non-empty set U a ultrafilter $\mathcal{A}$ is non-empty system of subsets of U s.t. (i): $0 \notin \mathcal{A}$, (ii): $A_1, A_2 \in \mathcal{A}$ then $A_1 \cap A_2 \in \mathcal{A}$, (iii): $A_1 \in \mathcal{A}$ and $A_1 \subseteq A_2$ then $A_2 \in \mathcal{A}$, and finaly

5. the space of positive hyperharmonic functions $\mathcal{H}^*(X)$ is linearly separating and there exists a strictly positive function $s_0 \in \mathcal{S}^*(X) \cap \mathcal{C}(X)$.

For better understanding of the definition reader can verify (or read e.g. in Bliedtner, Hansen [2]) that heat kernel in the classical Dirichlet problem is a harmonic kernel.

Finally we need to define Hunt process. As we will see from the definition it is a rather general stochastic process. However it still posses a Markov property.

Definition 3. A markov process $Z = (Z(t), t \geq 0)$ is called a Hunt process if the following conditions are satisfied:

1. the trajectories of $Z(t)$ are right-continuous on $t \geq 0$ and have limits from left for $0 \leq t \leq T$ for $T \in \mathbb{R}^+$
2. $(Z(t), t \geq 0)$ has the strong markov property w.r.t to the augmented natural filtration $(\mathcal{G}_t^Z, t \geq 0)$, i.e. given any $\mathcal{G}_t^Z$-adapted stopping time τ and $\forall s \geq 0, B \in \mathcal{B}(\mathbb{R}^d)$ the following holds:
$$P(Z(\tau + s) \in B | \mathcal{G}_t^Z) = P(Z(s) \in B | Z(\tau))$$
3. $(Z(t), t \geq 0)$ is quasi-left continuous, i.e. whenever (τ_n) is sequence of $\mathcal{G}_t^Z$-adapted stopping times and τ is $\mathcal{G}_t^Z$-adapted stopping time, s.t. $\tau_n \nearrow \tau$ then
$$\lim_{n \to \infty} Z(\tau_n) = Z(\tau) \text{ a.s. on } [\tau < \infty]$$

We will further denote by E_Z the set of all functions excessive with respect to Markov semigroup $\mathbb{P}$ of Hunt process Z.

3 Theorem

The following theorem provides us with a general result which allows us to describe the cone $\mathcal{W}$ of excessive functions for sub-Markov processes, where all excessive functions are lower semicontinuous and positive hyperharmonic functions for a family of harmonic kernels on X. Further $(X, \mathcal{W})$ is a balayge space.

Theorem 1. Let $\mathcal{P} \in \mathcal{C}^+(X)$ be a function cone and $\mathcal{W} = S(\mathcal{P}) := \{\sup f_n : (f_n) \in \mathcal{P} \textit{ increasing}\}$ s.t. $\mathbb{I} \in \mathcal{W}$. Then the following statements are equivalent:

(iv): $A_1 \in \mathcal{A}$ or $A_1^c \in \mathcal{A}$. Considering an ultrafilter $\mathcal{A}$ on U converging to point z from interior of U, then we say that ultrafilter $\mathcal{A}$ is regular w.r.t to the family of kernels H_U if $\lim_{x,\mathcal{A}} H_U(z, \cdot) = \delta_z$ (δ_z is dirac mass at z). Every ultrafilter which is not regular is called non-regular.

1. $(X, \mathcal{W})$ is a balayage space
2. there exists a family $(H_U)_{U\in\mathcal{U}}$ of harmonic kernels on X such that $\mathcal{H}^*(X) = \mathcal{W}$.
3. $\mathcal{W}$ is min-stable, i.e for every $f, g \in \mathcal{W}$ the infimum $\inf(f, g) \in \mathcal{W}$ and there exists a sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$ on X s.t. $E_{\mathbb{P}} = \mathcal{W}$
4. there exists a Hunt process $Z = (Z(t), t \geq 0)$ with state space $(X, \mathcal{X})$ s.t. $E_Z = \mathcal{W}$.

Proof: see Bliedtner J., Hansen W. (1986) [2], Theorem IV.8.1, p.168.

4 Application to Lèvy processes

Lèvy processes are an example of Markov processes with more restrictive conditions. They are also closely related to convolution semigroups of measures such that the distribution of the Lèvy process forms a convolution semigroup. To have nice insight about their behaving, we can also describe them as processes which can in general consist from the process with finite variation, a Brownian motion part and jump process which has countable many jumps. Each of these component can be missing so for example Brownian motion or Poisson process or mixture of these is Lèvy process.

Definition 4. A stochastic process $Z = (Z(t), t \geq 0)$ on X is a *Lèvy process* if the following conditions holds:

1. $Z(0) = 0$ a.s.
2. $(Z(t), t \geq 0)$ has independent and stationary increments
3. (stochastic continuity)
   $\forall \varepsilon > 0$ and all $h > 0$ $\lim_{h\to 0} \mathbb{P}(|Z(t + h) - Z(h)| > \varepsilon) = 0$
4. (cadlag property of trajectories)
   $(Z(t), t \geq 0)$ is right continuous in $t \geq 0$ and has left limits in $t > 0$.

The following proposition which we can formulate due to Theorem 1 gives us properties for the set of all excessive functions with respect to considered Lèvy process such that the set is closed under infimum and supremum and can be characterized via hyperharmonic functions.

Proposition 1. For a function cone $\mathcal{P} \in \mathcal{C}^+(X)$ and $\mathcal{W} = S(\mathcal{P})$ s.t. $\mathbb{I} \in \mathcal{W}$ and Lèvy process $Z = (Z(t), t \geq 0)$ with the state space $(X, \mathcal{X})$, s.t. $E_Z = \mathcal{W}$ there exists a family $(H_U)_{U\in\mathcal{U}}$ of harmonic kernels on X, s.t. $\mathcal{H}^*(X) = \mathcal{W}$, $\mathcal{W}$ is min-stable and $\sigma-$stable and there exists a sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$ on X, s.t. $E_{\mathbb{P}} = \mathcal{W}$

Proof: The fact that Lèvy process is Hunt process follows from definitions of these processes. Note, that Lèvy process is a Markov process (see e.g. Sato K. [4]) and every Lèvy process is a Feller process and every Feller process is a Hunt process (Applebaum D. [1]). The rest is a consequence of Theorem 1.

References

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